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Journal of APPLIED MATHEMATICS AND MECHANICS

www.elsevier.com/locate/jappmathmech

Journal of Applied Mathematics and Mechanics 69 (2005) 863–874

THE RECONSTRUCTION OF FEEDBACK USING OBSERVATION DATA†

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(*Received* 10 *August* 2004)

The problem of the restoration of an a priori unknown feedback which operates in a dynamical controlled system is considered. The restoration is achieved using results of observations during the motion of this system and approximate measurements of its actual phase positions. It is well known that this is an ill-posed problem. Two methods are proposed for solving it: a static method and a dynamic method. When solving the problem using the static method, the results of approximate measurements of the actual phase positions of the system in any given time interval serve as the input information. Here, restoration is achieved "a posteriori" when the corresponding time interval for the observation of the motion using the whole totality of admitted information expires. The concepts of the theory of preset control and the theory of ill-posed problems are invoked to solve the problem by this method. When the problem is solved by the dynamic method, the results of instantaneous approximate measurements of the actual phase positions of the system, which proceed to the observer in the dynamics during some specified time interval, serve as the input information for the solution. Here, the measurements and restoration are achieved in the dynamics over the course of the process using the real-time information. Concepts of the theory of positional control and the theory of dynamic regularization are invoked by the dynamic method. Constructive, stable, regularizing algorithms are built in order to solve the restoration problem by this as well as by the other method. Moreover, the dynamic algorithms are physically feasible and are capable of working under real-time conditions, processing the information which is being received during the course of the motion of the system and feeding the result into the dynamics as the motion develops.

Feedbacks in a dynamical system can be unknown a priori and must be determined (restored, reconstructed or identified) as a result of observations on the object. The restored feedback can then be used for operational control or more adequate modelling.

We will point out certain features of the static and dynamic approaches to the problem being considered. In the case of the static approach to solving the problem, the data for the calculation are known a priori, the restoration algorithm does not take account of any possible changes in these parameters during the calculation, and the calculation process itself is not, in general, a single process and it may be necessary to repeat it. However, in certain engineering and scientific developments, the need often arises to carry out restoration at the same time as the development of the process. In this case, the data for the calculations can only be received during the course of the process and now depends on how the restoration was carried out in the past. Similar problems are encountered in the mechanics of flight control and in problems of the operational development of information during the development of technological and production processes.

Problems of this kind for dynamical systems have been studied in different formulations in control theory, the game theory and the theory of estimation and identification [1–4]. The formulations with which we are concerned in this paper, as well as the methods for solving the problems are based from a conceptual point of view on the approaches of preset and positional control [1–7] and the approaches of the theory of ill-posed problems [8–10].

†*Prikl. Mat. Mekh.* Vol. 69, No. 6, pp. 962–975, 2005.
0021-8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.11.008

1. FORMULATION OF THE PROBLEM

We will now describe the information side of the problem and consider a dynamic control system, the behaviour of which in a specified bounded time interval $T = [t_0, \vartheta]$ $(-\infty < t_0 < \vartheta < +\infty]$ is described by the system of ordinary differential equations

$$\dot{x}(t) = f(t, x(t)) + u[t], \quad t_0 \le t \le \vartheta, \quad x(t_0) = x_0; \quad x(t), \quad u[t] \in R^n \tag{1.1}$$

where $x(t)$ is the state vector of the system at the instant of time $t \in T$ and $u[t]$ is the vector of the controlling action on the system at this instant of time. The action $u[t]$ is based on the feedback principle.

$$u[t] = A(t)x(t) + b(t), \quad t_0 \le t \le \vartheta$$

where $A(t)$ is a certain $n \times n$ matrix and $b(t)$ is a certain n-dimensional vector, defined for the instants of time $t_0 \le t \le \vartheta$.

Suppose the motion of the system is observed over a time interval T and the states of the system $x(t)$ are approximately measured at the corresponding instants of time $t \in T$ while, at the same time, the results of these measurements $y(t)$ satisfy the following criterion for the accuracy of the measurements

$$\|x(t) - y(t)\| \le \delta, \quad t \in T$$

where $\|\cdot\|$ is the Euclidean norm in R^n and δ is a numerical parameter which characterizes the accuracy of the measurements, $0 \le \delta \le \delta_0$.

The problem is as following. Using the results of the approximation measurements $y = y(t)$ of the motion of the system which is being observed $x = x(t)$, it is required to recreate approximately a form of the matrices $A = A(t)$ and the vectors $b = b(t)$ which determine the feedback and correspond to the results of observations. Here, the result $A_\delta = A_\delta(t)$ of the recreation of the matrices $A = A(t)$ and the result $b_\delta = b_\delta(t)$ of the recreation of the vectors $b = b(t)$ must be more accurate the smaller the errors in the measurements.

$$\int |||A(t) - A_\delta(t)|||^2 dt \to 0$$

$$\int \|b(t) - b_\delta(t)\|^2 dt \to 0, \quad \delta \to 0$$

Unless otherwise stated, all the functions are considered when $t \in T$ and integration with respect to t is carried out over the interval T; $|||\cdot|||$ is the Euclidean norm for the matrices $R^{n \times n}$. It is assumed that the matrices and vectors determining the feedback are unknown a priori; only certain a priori estimates of them are known the equations of the dynamics of the process and the initial state of the system are also known. We shall subsequently treat this problem as a problem concerning the reconstruction of a feedback.

The second aspect of the problem is associated with the restoration of an unknown feedback under conditions when measurements of the state of the system and the recreation of the required quantities determining the feedback must be carried out dynamically. Here, the problem consists of approximately recovering, throughout the course of the process, a form and the dynamics of the quantities $A = A(t)$, $b = b(t)$ determining the feedback which corresponds to the observation results using the results of approximate measurements of $y(t_i)$ at the corresponding discrete instants of time $t_i \in T$ of the actual states of the system $x(t_i)$ which become available to the observer. Here, the restoration must be more accurate the smaller the errors in the measurements and the more frequently the measurements of the states of the system are carried out, that is, the condition

$$\int |||A(t) - A_\delta^\Delta(t)|||^2 dt \to 0, \quad \int \|b(t) - b_\delta^\Delta(t)\|^2 dt \to 0, \quad \delta \to 0, \quad \Delta \to 0$$

must be satisfied by the results of the dynamic restoration of the required quantities $A_\delta^\Delta = A_\delta^\Delta(t)$ and $b_\delta^\Delta = b_\delta^\Delta(t)$, where Δ is the diameter of the partitioning of the interval T by the points t_i, for which

$$t_0 < t_1 < \ldots < t_i < \ldots < t_{m-1} < t_m = \vartheta$$

The diameter of the partitioning Δ will, as a rule, be chosen depending of the magnitude of the error in the measurements δ. In this dynamic formulation, it is also assumed that the matrices and vectors determining the feedback are unknown a priori, only certain a priori estimates of them are known and the equations of the dynamics of the process and the initial state of the system are also known.

We will now give a mathematical formulation of the problem. Suppose a function f is continuous in the set $T\times R^n$ and, in this set, satisfies the condition of sublinear growth and a local Lipschitz condition with respect to the variable x (see [5, 7, 11], for example). Suppose P is a convex, closed, bounded set of matrices from the space $R^{n\times n}$, U is the set of all measurable mappings $A=A(\cdot)\colon T\to P$, Q is a convex, bounded, closed set of vectors from the space R^n and V is the set of all measurable mappings $b=b(\cdot)$: $T\to Q$. The set of pairs of elements $w=(A,b)\in W=U\times V$ determines the set of permissible feedbacks in the problem being considered and we shall sometimes refer to the element $w\in W$ simply as the feedback. For each element $w\in W$, a unique solution $x(\cdot)=x(\cdot;w)=x(t;w)$ of the Cauchy problem (1.1) exists which is absolutely continuous in the interval T. We shall sometimes call this solution the motion of the dynamical system (1.1) which is generated by the feedback $w\in W$.

We now introduce the set of all possible solutions of the Cauchy problem (1.1) corresponding to the feedbacks $w\in W$

$$X=\{x(\cdot)=x(\cdot;w):w\in W\}$$

For each motion $x(\cdot)\in X$, we introduce the set of all permissible feedbacks corresponding to the given motion

$$W(x(\cdot))=\{w\in W:x(\cdot)=x(\cdot;w)\}$$

and the set of all possible measurements of this motion

$$Y(x(\cdot),\delta)=\{y\in L_2(T;R^n):\|x(t)-y(t)\|\le\delta\}$$

The problem consists of constructing an algorithm which, using any permissible measurements of the states of the observed motion of the system, approximately recovers the feedback of the system, which is in accord with the results of the observations of the motion. We identify the required algorithm with the family of mappings (methods)

$$D=\{D_\delta:0\le\delta\le\delta_0\},\quad D_\delta:L_2(T;R^n)\to E=L_2(T;R^{n\times n})\times L_2(T;R^n)$$

The initial problem can now be formulated as follows: it is required to construct an algorithm $D=\{D_\delta:0\le\delta\le\delta_0\}$ which, for any observed motion $x(\cdot)\in X$, possesses the regularizing property:

$$r_\delta(x(\cdot))\to 0,\quad \delta\to 0$$

$$r_\delta(x(\cdot))=\sup\{\rho[D_\delta(y),W(x(\cdot))]:y\in Y(x(\cdot),\delta)\}$$

$$\rho[D_\delta(y),W(x)]=\min\{\|D_\delta(y)-w\|_E:w\in W(x)\}$$

Before proceeding to solve this problem, we will point out several algebraic and topological properties of the motions of a system and the sets which have been introduced into the treatment. The set W is convex, bounded and closed, and it is therefore weakly compact in the space E, X is compact in the space $C(T;R^n)$, for each $x(\cdot)\in X$ the set $W(x(\cdot))$ is non-empty, convex, bounded and closed, and it is therefore weakly compact in the space E and has a unique element $w_*(x(\cdot))$ of minimum E-norm; if $w_k\rightharpoonup w$ weakly in E, then the strong convergence of the motions $x_k(\cdot)=x_k(\cdot;w_k)\to x(\cdot)=x(\cdot;w)$ in the space $C(T;R^n)$ holds and all the more in the space $L_2(T;R^n)$. It follows from this last property, in particular, that the mapping

$$E\supset W\ni w\to x(\cdot;w)\in X\subset L_2(T;R^n)$$

is completely continuous and therefore cannot have a continuous inverse mapping even if it is considered as a multiple-valued mapping. The ill-posed nature of the restoration problem follows from this, and also the need to use regularization methods to solve it. All of the numerical quantities and spaces considered in this paper are assumed to be real, mensurability and integrability are understood in the Lebesgue sense, and the definitions of the functional spaces used can be found in [11, 12], for example.

2. SOLUTION OF THE RESTORATION PROBLEM BY THE STATIC METHOD

We will construct the required algorithm. For any $\delta \in [0, \delta_0], y \in L_2(T; R^n)$, we define a realization of the method $D\delta(y)$ using the rule

$$D_\delta(y) = \eta \in W : F^*_\alpha(y) \le F_\alpha(\eta; y) \le F^*_\alpha(y) + \varepsilon \tag{2.1}$$

$$F^*_\alpha(y) = \min\{F_\alpha(s; y) : s \in W\}, \quad F_\alpha = F_\alpha(\eta; y) = \int\|x(t; \eta) - y(t)\|^2 dt + \alpha\|\eta\|^2_E \tag{2.2}$$

where $\alpha = \alpha(\delta)$ is a positive regularization parameter of the problem, $\varepsilon = \varepsilon(\delta)$ is a non-negative parameter, which characterizes the accuracy of the solution of extremal problem (2.1) and it will also be a regularization parameter.

Theorem 1. Suppose the regularization parameters $\alpha = \alpha(\delta)$ and $\varepsilon = \varepsilon(\delta)$ satisfy the following compatibility conditions

$$(\varepsilon(\delta) + \delta^2)\alpha(\delta)^{-1} \to 0, \quad \varepsilon(\delta) \to 0, \quad \alpha(\delta) \to 0, \quad \delta \to 0$$

The, the algorithm D, consisting of the methods (2.1), solves the restoration problem, that is, for any observed motion $x(\cdot) \in X$ when $\delta \to 0$, the convergence $r_\delta(x(\cdot)) \to 0$ holds and, furthermore, the strong convergence $\eta_\delta \to w_*(x(\cdot))$ in E holds for the realizations of the algorithm $\eta_\delta = D_\delta(y_\delta)$ when $\delta \to 0$ whatever realization of the measurements $y_\delta \in Y(x(\cdot), \delta)$ is the case here.

Proof. We specify an arbitrary element $x(\cdot) \in X$ and any dependences $\alpha = \alpha(\delta)$ and $\varepsilon = \varepsilon(\delta)$ which satisfy the condition of the theorem. For prove the theorem, it is sufficient to show that whatever the numerical sequence $\{\delta_k\} \subset [0, \delta_0]$, $\delta_k \to 0$ and the sequence of elements $\{y_k\}$, $y_k \in Y(x(\cdot), \delta_k)$, $k \in \{1, 2, 3, \ldots\}$, the following convergence holds

$$\rho[D_{\delta_k}(y_k), w_*(x(\cdot))] \to 0, \quad k \to \infty \tag{2.3}$$

Suppose the sequences $\{\delta_k\}$ and $\{y_k\}$ described above are chosen and fixed. We will now show that relations (2.3) hold. Taking account of the definition of the elements $\eta_k = D_{\delta_k}(y_k) \in W$, we can write the chain of inequalities

$$\alpha(\delta_k)\|\eta_k\|^2_E \le F_{\alpha(\delta_k)}(\eta_k; y_k) \le F^*_{\alpha(\delta_k)}(y_k) + \varepsilon(\delta_k) \le F_{\alpha(\delta_k)}(w_*(x(\cdot)); y_k) + \varepsilon(\delta_k) \le$$

$$\le \int\|x(t) - y_k(t)\|^2 dt + \alpha(\delta_k)\|w_*(x(\cdot))\|^2_E + \varepsilon(\delta_k) \le (\vartheta - t_0)\delta_k^2 + \alpha(\delta_k)\|w_*(x(\cdot))\|^2_E + \varepsilon(\delta_k)$$

from which we obtain the inequality

$$\|\eta_k\|^2_E \le (\vartheta - t_0)\delta_k^2\alpha(\delta_k)^{-1} + \varepsilon(\delta_k)\alpha(\delta_k)^{-1} + \|w_*(x(\cdot))\|^2_E \tag{2.4}$$

The boundedness of the sequence $\{\eta_k\}$ in the reflexive Banach space E follows from this inequality by virtue of the choice of regularization parameters, and it is therefore possible to separate out from the sequence $\{\eta_k\}$ a subsequence which weakly converges to a certain element $\eta^* \in E$. Without loss of generality, we can assume that this sequence itself converges weakly in E to the element η^*.

We will show that $\eta_k \to \eta^*$ strongly in E and that $\eta^* = w_* = w_*(x(\cdot))$. The following chain of inequalities

$$\int\|x(t; \eta^*) - x(t; w_*)\|^2 dt = \liminf\int\|x(t; \eta_k) - y_k(t)\|^2 dt \le \liminf F_{\alpha(\delta_k)}(\eta_k; y_k) \le$$

$$\le \limsup F_{\alpha(\delta_k)}(\eta_k; y_k) \le \limsup[(\vartheta - t_0)\delta_k^2 + \varepsilon(\delta_k) + \alpha(\delta_k)\|w_*\|^2_E] = 0, \quad k \to \infty$$

holds from which the following equality and inclusion are obtained

$$\int\|x(t; \eta^*) - x(t; w_*)\|^2 dt = 0, \quad \eta^* \in W(x(\cdot)) \tag{2.5}$$

From relations (2.4) and (2.5), we additionally obtain

$$\|w_*\|_E^2 \le \|\eta^*\|_E^2 \le \liminf\|\eta_k\|_E^2 \le \limsup\|\eta_k\|_E^2 \le \|w_*\|_E^2, \quad k \to \infty$$

whence we conclude that

$$\lim\|\eta_k\|_E^2 = \|\eta^*\|_E^2 = \|w_*\|_E^2, \quad k \to \infty$$

Then, by virtue of the uniqueness of the element of minimum E-norm in the set $W(x(\cdot))$, we obtain the equality $\eta^* = w_*$ and, by virtue of the reflexivity of the Banach space E, we obtain the strong convergence $\eta_k \to w_*$ in the space E from the weak convergence of the elements and the convergence of their norms. Hence, the convergence (2.3) holds.

Any minimization method can be used to solve extremal problem (2.1). As an example, we shall use the gradient projection method (see [13, 14], for example). We initially calculate the gradient of the functional (2.2), assuming that the components of the vector function f are continuously differentiable with respect to x in the set $T \times R^n$.

Since an element $y \in Y(x(\cdot), \delta)$ in the functional $F_\alpha(w; y)$ will subsequently remain fixed, for brevity we shall denote this functional simply by $F_\alpha(w)$. We now fix an arbitrary element $w = (A, b) \in W$ and add an arbitrary increment $h = (A_h, b_h) \in E$ to it. The difference in the motions

$$z(\cdot) = x(\cdot; w + h) - x(\cdot; w)$$

then satisfies the Cauchy problem

$$\begin{aligned}
\dot{z}(t) &= J(t, x(t))z(t) + G_1(x(t), z(t)) + G_2(x(t), z(t)), \quad t \in T, \quad z(t_0) = 0\\
G_1(x(t), z(t)) &= f(t, z(t) + x(t)) - f(t, x(t)) - J(t, x(t))z(t)\\
G_2(x(t), z(t)) &= A(t)z(t) + A_h(t)z(t) + A_h(t)x(t) + b_h(t)
\end{aligned}$$

where $x(\cdot) = x(\cdot; w)$, and $J(t, x(t))$ is the Jacobian of the vector function $f(t, x)$ with respect to x.

The increment of the functional can be represented in the form

$$\begin{aligned}
F_\alpha(w + h) - F_\alpha(w) &= 2I_1 + I_2 + 2\alpha\langle w, h\rangle_E + \alpha\|h\|_E^2\\
I_1 &= \int\langle x(t) - y(t), z(t)\rangle dt, \quad I_2 = \int\|z(t)\|^2 dt
\end{aligned}$$

The first integral in this equality can be converted to the form

$$\begin{aligned}
2I_1 &= I_{11} + I_{12} + I_{13}\\
I_{11} &= \int\langle\psi(t), A_h(t)x(t) + b_h(t)\rangle dt\\
I_{12} &= \int\langle\psi(t), A_h(t)z(t)\rangle dt, \quad I_{13} = \int\langle\psi(t), G_1(x(t), z(t))\rangle dt
\end{aligned}$$

where $\psi(\cdot) = \psi(\cdot; w)$ is the solution of the following linear problem, which we shall henceforth call the adjoint of problem (1.1)

$$\dot{\psi}(t) = -J^*(t, x(t))\psi(t) - A^*(t)\psi(t) - 2(x(t) - y(t)), \quad \psi(\vartheta) = 0$$

An asterisk on a matrix denotes the matrix which is the adjoint of it.

From the conditions which the parameters of the problem satisfy, it follows that a constant $C > 0$ exists such that the inequalities

$$I_2 \le C\|h\|_E^2, \quad I_{12} \le C\|h\|_E^2, \quad I_{13} < C\|h\|_E^2$$

hold for any motions $x(\cdot) \in X$, feedbacks $w \in W$, numbers $\delta \in [0, \delta_0]$ and measurements $y \in Y(x(\cdot), \delta)$.

The linear part of the increment in the functional can be represented in the form

$$I_{11} = \int \langle\langle A[\psi(t), x(t)], A_h(t)\rangle\rangle dt + \int \langle \psi(t), b_h(t)\rangle dt = \langle g, h\rangle_E$$

$$g = (A[\psi, x], \psi) \in E, \quad A[\psi, x] = A[\psi(t), x(t)]$$

where $\langle\langle \cdot, \cdot\rangle\rangle$ is a scalar product in $R^{n\times n}$ and $A[\psi(t), x(t)]$ is the contraction of the two vectors $\psi(t)$ and $x(t)$ into a matrix with elements

$$A[\psi(t), x(t)]_{ij} = \psi_i(t)x_j(t), \quad i, j \in \{1, \dots, n\}$$

From the estimates and representations which have been found, we obtain

$$F_\alpha(w+h) - F_\alpha(w) = \int \langle g + 2\alpha w, h\rangle_E dt + o(\|h\|_E)$$

This means that the functional F_α is Frechet differentiable at each point and

$$F_\alpha'(w) = g + 2\alpha w \in E$$

and

$$|o(\|h\|_E)| \le C_\alpha \|h\|_E^2, \quad C_\alpha = 3C + \alpha$$

Calculation of the gradient $F_\alpha'(w)$ reduces to the sequential execution of the following actions: solving the direct problem and finding its solution $x(\cdot) = x(\cdot; w)$, solving the adjoint problem and finding its solution $\psi = \psi(\cdot; w)$, calculating the matrix $A[\psi, x]$ and constructing the element $g + 2\alpha w$, which is also the required gradient.

We note several general properties of the gradient and of the minimization problem

$$F_\alpha(w) \to \min : w \in W \tag{2.6}$$

The gradient satisfies the Lipschitz condition in W, and its non-linear part $F_0'(w) = g$ is the weakly-strong continuous mapping $E \supset W \to E$. The functional F_α is bounded in W; any set of it of the level

$$M_\alpha(z) = \{w \in W : F_\alpha(w) \le F_\alpha(z)\}$$

is bounded and weakly compact in E; it is weakly semicontinuous from below in W and reaches its minimum value

$$F_\alpha^* = \min\{F_\alpha(w) : w \in W\}$$

which is non-negative and finite, in W; the set of all elements

$$W_\alpha^* = \{w \in W : F_\alpha(w) = F_\alpha^*\}$$

which minimize the functional F_α is non-empty and weakly compact in E; any minimizing sequence of problem (2.6) converges strongly in E to the set W_α^*; for any $z \in W$ the set

$$S_\alpha^*(z) = \{w \in M_\alpha(z) : \langle F_\alpha'(w), v - w\rangle_E \ge 0 \quad \forall v \in W\}$$

of all stationary points of the functional from a set of level $M_\alpha(z)$ is non-empty and weakly compact in E and, if, for any $z \in W$, the set

$$S_\alpha^0(z) = \{w \in M_\alpha(z) : F_\alpha'(w) = 0\}$$

is non-empty, then it is compact in E.

We will now consider the iterative process of the gradient projection method ($k = 0, 1, 2, \dots$) for minimization problem (2.6)

$$w_{k+1} = \Pr(w_k - \gamma_k F_\alpha'(w_k)), \quad w_0 \in W, \quad \sigma_1 \le \gamma_k \le 2/(L + 2\sigma_2)$$

where σ_1 and σ_2 are positive numbers which are the parameters of the method, L is the Lipschitz constant of the gradient F'_α in the set W and Pr is a projection operator in W (the projection exists and is unique).

Theorem 2. Whatever the initial approximation $w_0 \in W$, the sequence $\{w_k\}$ of the gradient projection method is a relaxation sequence and weakly converges in E to the set $S^*_\alpha(w_0)$. Furthermore, the following convergences hold

$$\langle F'_\alpha(w_k), w_{k+1} - w_k \rangle_E \to 0, \quad \|w_{k+1} - w_k\|_E \to 0, \quad k \to \infty$$

If the inclusion $z^* \in W$ holds for a certain $w_0 \in M_\alpha(z^*) \subseteq W$, then we additionally have the strong convergence in E

$$F'_\alpha(w_k) \to 0, \quad w_k \to S^0_\alpha(w_0), \quad k \to \infty$$

Moreover, if the inequality

$$\|F'_\alpha(w)\|_E \geq d(F_\alpha(w) - F^*_\alpha)$$

is satisfied for a certain constant $d > 0$ in the elements $w \in M_\alpha(w_0)$, then the sequence $\{w_k\}$ is a minimizing sequence and the following estimate of the rate of convergence for the functional holds

$$0 \leq F_\alpha(w_k) - F^*_\alpha \leq C^* k^{-1}, \quad k = 1, 2, \ldots, \quad C^* = \text{const} \geq 0$$

The proof of the theorem is analogous to the proof of similar assertions [13, Theorems 8.4.1 and 8.4.2].

3. SOLUTION OF THE RESTORATION PROBLEM BY THE DYNAMIC METHOD

The corresponding justifications and various examples of interesting inverse problems in which it is important to obtain dynamic solutions are presented, for example, in [1, 15–30]. In the required algorithm D, which must solve the restoration problem, we identify each method D_δ with a family of mappings

$$D_\delta = \{D^t_\delta : t_0 \leq t \leq \vartheta\}, \quad D^t_\delta : R^n \times R^n \to P \times Q \tag{3.1}$$

We call the function $w_\delta = w_\delta(\cdot; y) : [t_0, \vartheta] \to P \times Q$ which is defined by the equality

$$w_\delta(t) = D^t_\delta(y(t), z(t))$$

the realization of algorithm D for the measurement $y \in Y(x(\cdot), \delta)$ and we denote it by the symbol $D_\delta(y)$. Sometimes, we shall detail this notation $D_\delta(y) = (A_\delta(\cdot), b_\delta(\cdot))$. Here, the variable z is an internal variable of the algorithm. Its value $z = z(t)$ at the instant of time t is uniquely formed on the basis of the permissible information $y(\tau)$, $t_0 \leq \tau \leq t$ concerning the motion of the system which has been accumulated up to this instant of time. We will formulate a rule for forming the variable $z = z(t)$ below when we consider the actual method for constructing the algorithm.

The initial problem can now be formulated as follows: it is required to construct an algorithm D which consists of the methods (3.1) and which, for any observed motion $x(\cdot) \in X$, possesses the regularizing property $r_\delta(x(\cdot)) \to 0$, $\delta \to 0$.

We will now construct the algorithm which solves the above problem. To construct the required algorithm we will make use of the method of dynamic regularization with the model described earlier in [1, 15]. Such a hypothetical unit as a model of the initial system will participate in the constructions. Using this model, the values of an auxiliary internal variable for the corresponding algorithm will be formed. However, this hypothetical model can be implemented quite practically on a computer.

For any $t \in T$, $\delta \in [0, \delta_0]$, $y \in R^n$, $z \in R^n$, we define the mappings D^t_δ at a point (y, z) according to the rule

$$D^t_\delta(y, z) = \eta = (A_\eta, b_\eta) \in P \times Q$$

$$H(\eta) \leq \min\{H(s) : s = (A_s, b_s) \in P \times Q\} + \varepsilon(\delta) \tag{3.2}$$

$$H(s) = 2\langle z - y, A_s y + b_s \rangle + \alpha(\delta)(|||A_s|||^2 + \|b_s\|^2)$$

where $\varepsilon = \varepsilon(\delta)$ and $\alpha = \alpha(\delta)$ are positive regularization parameters.

We define the value $z(t)$ of the internal variable z for the instant of time $t \in T$ as the value at this instant of the solution of the Cauchy problem for the system-model

$$\dot{z}(\tau) = f(\tau, z(\tau)) + A_\delta(\tau)z(\tau) + b_\delta(\tau), \quad t_0 \le \tau \le t, \quad z(t_0) = x_0$$

The solution of this Cauchy problem, from a practical computational point of view, is conveniently carried out using a discrete scheme which is similar to the way that ordinary differential equations are solved using the Euler scheme [7]. In this connection, we shall describe the operation of the algorithm of restoration in the dynamics in a scheme which is discrete in time.

Any relations $\varepsilon = \varepsilon(\delta)$ and $\alpha = \alpha(\delta)$ and any partitioning of T into intervals $[t_0, t_1), \dots, [t_i, t_{i+1}), \dots, [t_{m-1}, \vartheta]$ by the points $t_i : t_0 < t_1 < \dots < t_m = \vartheta$ are initially specified. The diameter of this partitioning will be chosen depending on the value of the accuracy of the measurements δ, $\Delta = \Delta(\delta)$.

We will now describe the step-by-step construction of the realization of the algorithm. Suppose an observation is made for any motion $x(\cdot) \in X$. Any feedback $w = (A, b) \in W(x(\cdot))$ is subject to reconstruction.

Step $i = 0$. At the instant of time t_0, information comes to the observer in the form of a measurement $y(t_0)$ of the state of motion $x(t_0)$. By putting $y = y(t_0)$ and $z = y(t_0)$, the observer determines that realization of the method

$$D_\delta^{t_0}(y, z) = (A_\delta(t_0), b_\delta(t_0))$$

at the instant of time t_0 according to rule (3.2).

The function

$$w_\delta^{(0)}(t, \cdot) = D_\delta^{t_0}(y, z), \quad t_0 \le t < t_1$$

which is constant in time, is taken as an approximation to the required feedback w in the time interval $t_0 \le t < t_1$. The Cauchy problem for the system-model

$$\dot{z}(\tau) = f(\tau, z(\tau)) + A_\delta(t_*)z(\tau) + b_\delta(t_*), \quad t_* \le \tau \le t^*, \quad z(t_*) = z \tag{3.3}$$

is then solved for the interval $[t_*, t^*] = [t_0, t_1]$ and the state $z(t^*)$ of its solution is stored.

Step $i = 1$. At the instant of time t_1, information is fed to the observer in the form of a measurement $y(t_1)$ of the state of the motion $x(t_1)$. Putting $y = y(t_1)$ and $z = z(t_1)$, the observer determines the realization of the method

$$D_\delta^{t_1}(y, z) = (A_\delta(t_1), b_\delta(t_1))$$

according to rule (3.2).

The function

$$w_\delta^{(1)}(t, \cdot) = D_\delta^{t_1}(y, z), \quad t_1 \le t < t_2$$

which is constant in time, is taken as the approximation for the required feedback w in the time interval $t_1 \le t < t_2$. The Cauchy problem (3.3) is then solved for the interval $[t_*, t^*] = [t_1, t_2]$ and the state $z(t^*)$ of the solution is stored.

The following steps for $i = 2, \dots, m-1$ are analogous to the step $i = 1$. Hence, a piecewise-constant in time realization of the method

$$D_\delta(y) = w_\delta(t) = w_\delta^{(i)}(t), \quad t_i \le t < t_{i+1}, \quad i = 0, \dots, m-1 \tag{3.4}$$

will be obtained successively throughout the course of the process (in the dynamics) up to the final instant of time $t_m = \vartheta$.

For the description of the operation of the algorithm in time it is clear that it can be implemented under real-time conditions.

Theorem 3. Suppose the regularization parameters $\alpha = \alpha(\delta)$ and $\varepsilon = \varepsilon(\delta)$ and the magnitude of the diameter $\Delta = \Delta(\delta)$ of the partitioning of the time interval T satisfy the following compatibility condition

$$(\varepsilon(\delta)+\delta^2+\Delta(\delta)^{1/2})\alpha(\delta)^{-1} \to 0, \quad \varepsilon(\delta)\to 0, \quad \alpha(\delta)\to 0 \quad \text{when} \quad \delta \to 0$$

The algorithm D, consisting of the methods (3.4), then solves the restoration problem, that is, the convergence $r_\delta(x(\cdot)) \to 0$ holds for any observed motion $x(\cdot) \in X$ when $\delta \to 0$ and, moreover, the strong convergence in E $w_\delta \to w_*(x(\cdot))$ holds for the realizations of the algorithm $w_\delta = D_\delta(y_\delta)$ when $\delta \to 0$ whatever realizations of the measurements $y_\delta \in Y((x(\cdot)), \delta)$ occur here.

Proof. We fix an arbitrary element $x(\cdot) \in X$ and any relations

$$\alpha = \alpha(\delta), \quad \varepsilon = \varepsilon(\delta), \quad \Delta = \Delta(\delta)$$

which satisfy the condition of the theorem. For prove the theorem, it is sufficient to show that, whatever the numerical sequence $\{\delta_k\} \subset [0, \delta_0]$, $\delta_k \to 0$ and the sequence of elements $\{y_k\}$, $y_k \in Y(x(\cdot), \delta_k)$, $k \in \{1, 2, 3, \dots\}$, the following convergence holds

$$\rho[D_{\delta_k}(y_k), w_*(x(\cdot))] \to 0, \quad k \to \infty; \quad w_* = w_*(x(\cdot)) = (A_*, b_*) \tag{3.5}$$

We will now fix and sequences $\{\delta_k\}$ and $\{y_k\}$ which satisfy the above-mentioned conditions and show that relations (3.5) hold.

Taking account of the rule for the formation of a realization of the algorithm

$$w_k = D_{\delta_k}(y_k) = (A_k, b_k)$$

the following estimate for the functional Λ_k can be obtained

$$\Lambda_k(t) = \|x(t)-z_k(t)\|^2 + \alpha(\delta_k)\int_{t_0}^{t}\Omega_k(\tau)d\tau \le \nu_k$$

$$\Omega_k(t) = |||A_k(t)|||^2 + \|b_k(t)\|^2 - |||A_*(t)|||^2 - \|b_*(t)\|^2$$

$$\nu_k = C_*[\varepsilon(\delta_k)+\delta_k^2+\Delta(\delta_k)^{1/2}]$$

where C_* is a certain positive constant which is independent of k and is only determined a priori by the known data on the system and the problem; z_k is the motion of the system-model which corresponds to the feedback w_k (The construction of this motion has been described in detail above.)

From the above estimate we obtain

$$\max\{\|x(t)-z_k(t)\|^2 : t \in T\} \le \nu_k + 2\alpha(\delta_k)(\vartheta - t_0)\Omega_0 \tag{3.6}$$

$$\begin{gathered}\Omega_0 = \max\{|||A|||^2 + \|b\|^2 : A \in P, b \in Q\} \\ \|w_k\|_E^2 \le \|w_*\|_E^2 + \nu_k\alpha(\delta_k)^{-1}\end{gathered} \tag{3.7}$$

Taking account of the weak compactness of the set W in the Hilbert space E, we can assume without loss in generality that, for a certain element $w^* \in W$,

$$w_k \rightharpoonup w^* \text{ weakly in } E \tag{3.8}$$

Then, for any $t \in T$, we have the convergence in R^n

$$z_k(t) = z_k(t; w_k) \to x(t) = x(t; w^*) \tag{3.9}$$

From relations (3.6) and (3.9), we then obtain the equalities $x(t; w_*) = x(t; w^*)$, from which it follows that $w^* = w_*$. From the properties (3.7) and (3.8) and the last equality, we obtain the chain of inequalities

$$\|w^*\|_E \le \liminf \|w_k\|_E \le \limsup \|w_k\|_E \le \|w^*\|_E, \quad k \to \infty$$

from which the convergence of the norms

$$\|w_k\|_E \to \|w^*\|_E$$

in the Hilbert space E follows.

If the weak convergence (3.8) is now taken into account, we obtain the convergence

$$w_k \to w_* \text{ strongly in } E$$

Hence, the convergence (3.5) holds.

4. REMARKS

1. The constraints on the parameters of the problem also ensure the convergence of the approximations of the feedback which have been found in the space $L_q(T; R^{n\times n}) \times L_q(T; R^n)$ for any $q \in [1, \infty)$. This follows from the convergence of the approximations in E and the boundedness of the set W in $L_\infty(T; R^{n\times n}) \times L_\infty(T; R^n)$.

2. The results obtained also hold for a non-linear feedback of the form

$$u[t] = A(t)\varphi(t, x(t)) + b(t), \quad t_0 \le t \le \vartheta$$

where φ is a certain known function with the same properties as the function f. At the same time, when solving the restoration problem by the static method, the formulation of Theorems 1 and 2 remains the same, and it is only necessary, when calculating the gradient of the functional (2.2) in the adjoint problem, to replace the expression $A^*(t)\psi(t)$ by the expression $J_\varphi^*(t, x(t))A^*(t)\psi(t)$, where $J_\varphi(t, x)$ is the Jacobian of the vector function $\varphi = \varphi(t, x)$ with respect to the variable x.

In solving the restoration problem by the dynamic method, the formulation of Theorem 3 remains as before, and it is only necessary to replace the term $A_s y$ in the expression for the functional H by the term $A_s\varphi(t, y)$ and, on the right-hand side of the system-model, to replace the term $A_\delta(t)z(t)$ by $A_\delta(t)\varphi(t, z(t))$.

3. The regularizing algorithms for solving the restoration problem which have been constructed possess the property of uniform regularizability in the sets of motions corresponding to the compact sets of permissible feedbacks. Suppose W_* is a subset of W which is compact in E, and X_* is the set of all motions corresponding to the feedbacks from W_*. Then,

$$\sup\{r_\delta(x(\cdot)) : x(\cdot) \in X_*\} \to 0, \quad \delta \to 0$$

4. In the problem being considered, as in similar problems which have been solved in [1, 15–30], for example, corresponding estimates of the rate of convergence of the methods can be determined and the convergences in stronger metrics can also be found.

5. The results obtained here can easily be extended to extensive classes of problems in which the units are described by systems with distributed parameters (see [15–30], for example).

6. In applications, it is often required that the corresponding operator for solving an inverse problem should possess the property of physical feasibility: the restoration results synchronize until the information entering into the input synchronizes. We note that the operator for solving the direct problem possesses this property. The property of the dynamic restoration of the feedback also possesses this property. The property of the physical feasibility of an operator for solving an inverse problem is found to be extremely important in situations when the results of the restoration are used in feedback systems and must be used in the system then and there throughout the course of a process (such is the state of affairs in automatic control systems). We note that a posteriori methods for the solution of inverse problems in dynamics, among which gradient methods are very widespread, do not, as a rule, possess the property of physical feasibility.

5. EXAMPLE

We will now present the results of numerical simulation using the dynamic restoration of feedback $u[t] = a(t)x(t)$ in the dynamical system

$$\dot{x}(t) = x(t)\sin x(t) + u[t], \quad t_0 \le t \le \vartheta, \quad x(t_0) = x_0, \quad x \in R$$

Restoration of feedbacks with the following functions: $a(t)$, $t_0 \le t \le \vartheta$; $a(t) = 1 + \sin 2\pi t$ (a "smooth" feedback); $a(t) = t$ when $t_0 \le t \le t_1$; $a(t) = a_0 - t$ when $t_1 < t \le \vartheta$ (feedback with a "break"); $a(t) = a_1$ when $t_0 \le t \le t_1$ and $a(t) = a_2$ when $t_1 < t \le \vartheta$ ("discontinuous" feedback), was carried out in accordance with the method described above. In this case, the a priori information

$$P = [b_1, b_2], \quad -\infty < b_1 < b_2 < +\infty$$

was used for the feedback.

The interference in the measurements was simulated by the relation

$$y(t) = x(t) + \delta \sin pt, \quad p = \text{const}$$

The following correspondences of the regularization parameters were adopted

$$\varepsilon(\delta) = 0, \quad \alpha(\delta) = \delta^{1/3}, \quad \Delta(\delta) = \delta$$

The results of the calculations on the restoration of the above-mentioned feedbacks for the following values of the parameters of the problem

$$t_0 = 0, \quad \vartheta = 1, \quad x_0 = 1, \quad t_1 = 0.5, \quad a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \quad b_1 = 0, \quad b_2 = 3, \quad p = 1.$$

are shown in Fig. 1.

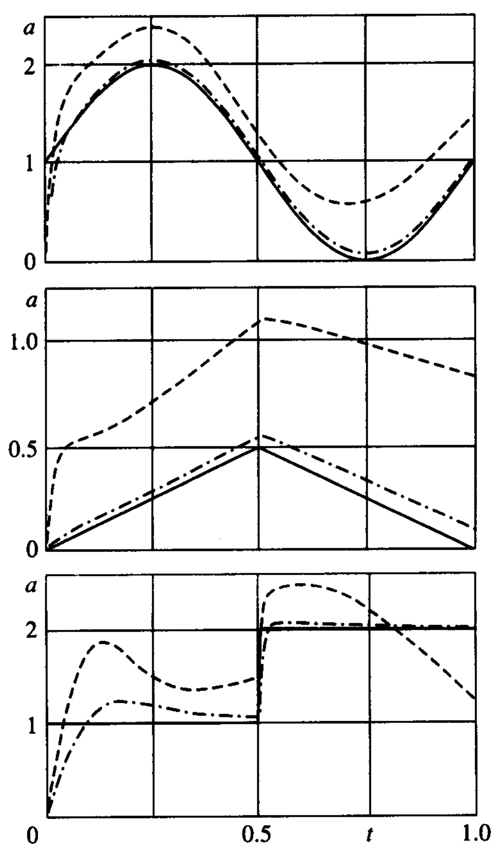

Fig. 1

The restoring function (feedback) $a = a(t)$, $t_0 \le t \le \vartheta$ is shown by the solid curve, the result of restoration when $\delta = 1$ by the dashed curve and, when $\delta = 0.1$, by the dot-dash curve.

This research was supported by the Russian Foundation for Basic Research (05-01-00098) and the Programme for Basic Research of the Praesidium of the Russian Academy of Sciences.

REFERENCES

1. OSIPOV, Yu. S. and KRYAZHIMSKII, A. V., *Inverse Problems of Ordinary Differential Equations: Dynamical Solutions*. Gordon and Breach, London, 1995.
2. KURZHANSKII, A. B., *Control and Observation under Conditions of Uncertainty*. Nauka, Moscow, 1977.
3. CHERNOUS'KO, F. L. and MELIKYAN, A. A., *Game problems of Control and Search*, Nauka, Moscow, 1978.
4. KRUT'KO, P. D., *Inverse Problems of the Dynamics of Controlled Systems. Non-Linear Models*. Nauka, Moscow, 1988.
5. PONTRYAGIN, L. S., BOLTYANSKII, V. G., GAMKRELIDZE, R. V. and MISHCHENKO, Ye, F., *Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1976.
6. KRASOVSKII, N. N., *The Theory of the Control of Motion*. Nauka, Moscow, 1968.
7. KRASOVSKII, N. N. and SUBBOTIN, A. I., *Position Differential Games*. Nauka, Moscow, 1974.
8. TIKHONOV, A. N. and ARSENIN, V. YA., *Methods of Solving Ill-Posed Problems*. Nauka, Moscow, 1979.
9. IVANOV, V. K., VASIN, V. V. and TANANA, V. P., *The Theory of Linear Ill-Posed Problems and its Application*. Nauka, Moscow, 1978.
10. LAVRENT'YEV, M. M., ROMANOV, V. G. and SHISHATSKII, S. P., *Ill-Posed Problems of Mathematical Physics and Analysis*. Nauka, Moscow, 1980.
11. WARGA, J., *Optimal Control of Differential and Functional Equations*. Academic Press, London, 1972.
12. YOSHIDA, K., Functional Analysis. Springer, Berlin, 1966.
13. VASIL'YEV, F. P., *Methods of Optimization*. Faktorial, Moscow, 2002.
14. IOFFE, A. D. and TIKHOMIROV, V. M., *The Theory of Extremal Problems*. Nauka, Moscow, 1974.
15. OSIPOV, Yu. S., VASIL'YEV, F. P. and POTAPOV, M. M., *Fundamentals of the Method of Dynamic Regulation*. Izd. MGU, Moscow, 1999.
16. KRYAZHIMSKII, A. V. and OSIPOV, Yu. S., Modelling of control in a dynamical system. *Izv. Akad. Nauk SSSR. Tekhn. Kibernetika*, 1983, 2, 51–60.
17. VDOVIN, A. Yu., Error estimates in the problem of restoration of control. In *Problems of Positional Modelling*. Ural'skii Nauch. Tsentr Akad Nauk SSSR, 1986, 3–11.
18. MAKSIMOV, V. I., The modelling of controls in parabolic variational inequalities. *Differents. Uravneniya*, 1991, **27**, 9, 1603–1609.
19. KOROTKII, A. I., Inverse problems of the dynamics of controlled systems with distributed parameters. *Izv. Vuzov. Matematika*, 1995, 11, 101–124.
20. KOROTKII, A. I., The restoration of the locations and intensities of sources of perturbations. *Tr. Inst. Matematiki i Mekhaniki Ural'skogo Otd. Ross. Akad. Nauk*, 1996, **4**, 219–2277.
21. KOROTKII, A. I. and TSEPELEV, I. A., Upper and lower error estimates in the problem of the dynamic determination of parameters. *Tr. Inst. Matematiki i Mekhaniki Ural'skogo Ord. Ross. Akad.Nauk*, 1996, **4**, 228–238.
22. KOROTKII, A. I., Restoration of a set of controls using measurements of the states of evolution of a system. *Prikl. Mat. Mekh.*, 1997, **61**, 3, 440–446.
23. KOROTKII, A. I., Restoration of the controls and parameters of dynamical systems in the case of incomplete information. *Izv. Vuzov. Matematika*, 1998, 11, 47–55.
24. KOROTKII, A. I. and TSEPELEV, I. A., On an inverse dynamic problem for Goursat–Darboux system. *J. Math. Systems, Estimation and Control*, 1998, **8**, 2, 181–184.
25. KOROTKII, A. I., Restoration of controls under conditions of incomplete information regarding the dynamics of the system. Prikl. Mat. Mekh., 1998, **62**, 4, 566–575.
26. KOROTKII, A. I., The dynamic reconstruction of controls and parameters under conditions of incomplete information on the system. Differents. Uravneniya, 1999, **35**, 11, 1482–1486.
27. KOROTKII, A. I., Dynamic restoration of controls under conditions of uncertainty. *Izv. Ross., Akad. Nauk. Teoriya i Sist. Uprav*. 2000, 1, 21–24.
28. ISMAIL-ZADEH, A. T., KOROTKII, A. I., NAIMARK, B. M. and TSEPELEV, I. A., Three-dimensional modelling of an inverse retrospective problem of thermal convection. *Zh. Vychisl. Mat. Mat. Fiz.*, 2003, **43**, 4, 614–626.
29. KOROTKII, A. I. and TSEPELEV, I. A., Solution of a retrospective inverse problem for a nonlinear evolutionary model. *Proc. Steklov Inst. Mathematics.*, Suppl. 2, 2003.
30. ISMAIL-ZADEH, A. T., SCHUBERT, G., TSEPELEV, I. A. and KOROTKII, A. I., Inverse problem of thermal convection: numerical approach and application to mantle plume restoration. *Phys. Earth and Planetary Interiors*, 2004, **145**, 1–4, 99–114.

Translated by E.L.S.